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MAY 1963

CONCEPTS AND THEORIES OF  
PURE COMPETITION

Lloyd S. Shapley and Martin Shubik

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*The* **RAND** *Corporation*  
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PREFACE

In this Memorandum the authors discuss a competitive situation arising in economic theory, using some concepts and techniques from the mathematical theory of games.

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## SUMMARY

A classical formulation of the marketplace is examined from three different theoretical viewpoints, and the three solutions that result are contrasted. When the number of participants is small, the solutions are very different, both in their form and in their quantitative predictions. As the size of the market increases, however, they all converge to a common solution, despite the wide disparity in the underlying assumptions.

The first model supposes all individuals to be price-takers, acting in isolation to maximize their private utility; the result is the "competitive equilibrium," or Walras solution. The second model is a fully cooperative game, with no restrictions on communication, negotiation, or collusion; the result is the "core," or Edgeworth solution. The third is an asymmetric, noncooperative game, in which half the players (e.g., the consumers) are passive price-takers, while the others have strategic freedom to exploit this passivity, individually but not collusively. The "equilibrium point" of this game corresponds to the oligopolistic solution of Cournot.

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## CONCEPTS AND THEORIES OF PURE COMPETITION

### 1. INTRODUCTION

Underlying much of basic economic theory concerning the formation of prices and the operation of markets is the idea that if the number of individuals trading in all markets is sufficiently large, and if there are no institutional bounds, or opportunities for cooperative arrangements between groups, then a competitive price system will emerge. Each individual may proceed to maximize his own welfare, utilizing only his knowledge of the price levels and of his own desires and assets, with the overall result that all markets will be cleared and the resulting imputation of goods and services will meet certain broad requirements of optimality.

At the other end of the spectrum Edgeworth [1], and many others since, have observed that in bargaining among a few individuals (of more or less comparable strength), there will be a whole range of outcomes that are optimal, in the appropriate sense, and economic conditions alone will not determine a specific outcome.

A third possibility is a situation in which individuals of one type are few in number, or are organized in some manner, while the others are not. In this case it has been observed that the few can employ "monopolistic practices" against the others, and that the resultant outcome will

be determinate, but not optimal. Institutionally, such a situation may arise when a few firms confront many customers in a market; the natural setting for this type of analysis is within the framework of the theory of oligopoly.

An interpretation and unification of these three different viewpoints is presented here in terms of the theory of games. The results described are based primarily upon the joint findings of Shapley and Shubik [2], but also upon recent papers by Shubik [3], Shapley [4], Scarf [5], Aumann [6], and Debreu [7].

## 2. BILATERAL MONOPOLY

### 2.1. The Edgeworth Model

For simplicity we consider a market with two traders, trading in two commodities. The initial holding of the first trader is  $(a, 0)$  and that of the second is  $(0, b)$ . Let their preferences be represented by two families of continuously differentiable, convex indifference curves, denoted by  $\psi(x, y)$  and  $\varphi(x, y)$ , respectively, where  $x$  and  $y$  are the amounts held of the first and second commodity. By superimposing the two families of curves, with coordinates oppositely oriented, we obtain the familiar "Edgeworth box" [1], as illustrated in Fig. 1. Any point in the box represents a jointly attainable trade and will have coordinates  $(a-x, b-y)$  for the first

trader and  $(x,y)$  for the second. The point  $R$  represents the initial position of both players, prior to trading. The first trader's zero point is at  $O'$ , and his initial holdings are represented by the vector  $O'R$ , which has length  $a$ . His goal is to carry out trades that move the outcome in a "south-west" direction as far as possible, i.e., away from  $O'$ . The zero point for the second trader is at  $O$ , and  $OR$  represents his initial stock of  $b$  units of the second commodity. He wishes to trade in a manner that moves the outcome in a "north-east" direction as far as he can.

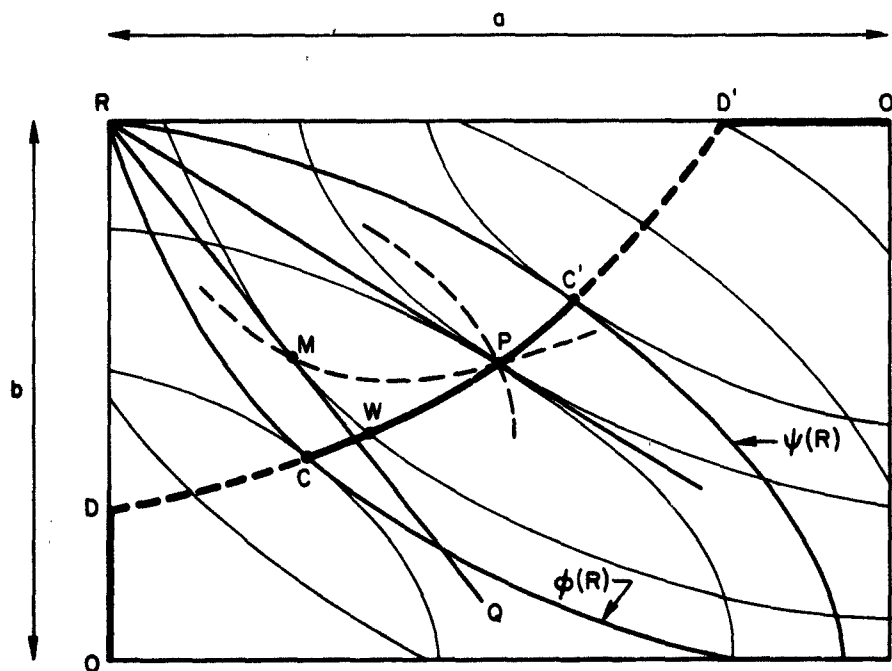


Fig. 1



There are three basic models, or "games," which can be formulated in bilateral monopoly to illustrate three strategically very different trading procedures; we shall denote them by the symbols  $(1, 1)_0$ ,  $(1, 1)_1$  and  $(1, 1)_2$ . In Fig. 1 they will lead to the outcomes indicated by the point P, the point M, and the curve CC', respectively.

## 2.2. The Game $(1, 1)_0$ : The Competitive Equilibrium

The essence of the economic model of pure competition is that all participants act as price-takers. One assumes that in some manner or other a schedule of prices has been established in the market, and that each individual takes note of these prices and buys or sells accordingly. He does not actively control or influence the prices. When the number of participants in the market is large, it is usually assumed either implicitly or explicitly that a dynamic market mechanism produces the prices, but in our "game," which at present has just two players, we shall unrealistically assume that the omniscient referee performs his calculations and names an appropriate set of prices, then acts as a clearing house for all transactions. By the rules of the game, the players are strategically constrained to act as price-takers. The subscript on the symbol  $(1, 1)_0$  is meant to convey that neither of the two traders has sufficient freedom of strategy to manipulate price.

We know from basic economic theory [8] that when consumers' tastes are independent and can be represented by convex indifference sets (as is the case here), then a competitive equilibrium will always exist. This is illustrated in Fig. 1 by the equilibrium price line  $RP$  and the competitive allocation point  $P$ . In this two-commodity example, the slope of the line  $RP$  gives the relative price of one commodity in terms of the other. If prices in this ratio are announced by the referee, then trading will continue until the point  $P$  is reached. No further gain can be made by either side because of the tangency between the price line and each trader's indifference curve. This so-called "competitive" solution exists under quite general conditions, but it is not necessarily unique.\*

The point  $P$  is Pareto optimal, which means that, given the final distribution of resources, it is not possible to improve the welfare of any one individual without decreasing the welfare of another. The extended curve  $ODD'O'$  describes the full Pareto-optimal set in Fig. 1. (The interior portion  $DD'$  is the locus of the

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\*Somewhat unfortunately, the term "competitive" has been attached by established usage to just that solution concept—out of the three or four that we shall be considering—that has the least to do with game theory. Paradoxically, the competitive equilibrium involves no interplay between the "competitors," and might better be described as an "administered-price" solution.

points of tangency between the two sets of indifference curves.) Any point off this curve can be dominated by a point on it. For instance, M and W in Fig. 1 are indifferent to the second trader, but W is preferred to M by the first trader. Hence M is not Pareto optimal, since the welfare of at least one trader can be improved without damage to the other.

### 2.3. The Game $(1, 1)_1$ : The Monopolistic Solution

The second case we consider is  $(1, 1)_1$ , where one of the traders, say the first, is strategically empowered to name the price, while the other is restricted to maximizing his own welfare, taking price as given. Any ray through the initial point R represents a set of (relative) prices. The object of the first trader will be to select a ray such that the final trade is as favorable as possible to himself, knowing that the second player will trade up to the point of tangency between the price ray and his family of indifference curves. This is indicated in Fig. 1 for the ray RQ at the point M. A monopolistic trading curve may be drawn—the curve MP in the figure—which is the locus of points of tangency between the price rays and the indifference curves of the second trader. (There will be a similar "response curve" for the first player, intersecting the other at P.) The optimum for the first player is in

fact the point M, where his indifference curve is tangent to the second player's response curve.

We see that this "monopolistic" solution is not Pareto optimal, and that it favors the price-naming monopolist as compared to the solution P of the previous case.

#### 2.4. The Game $(1, 1)_2$ : The Contract Curve

In the third case, denoted by  $(1, 1)_2$ , we assume that both players are strategically capable of naming price, and may negotiate in any manner they choose. There is no specification of the dynamic process—the particular "rules of the game"—beyond noting that any bargaining or haggling is permitted. The solution propounded by Edgeworth to this model of bilateral monopoly was the contract curve CC' in Fig. 1. This is not the entire Pareto-optimal set, but just that part of it where neither side can force a more favorable distribution by a refusal to trade. The two indifference curves through R provide the bounds C and C'.

This Edgeworth solution does not yield a unique prediction of the imputation of resources; it merely indicates a range. The price-taking model  $(1, 1)_0$ , on the other hand, did produce a determinate, Pareto-optimal outcome; the latter was located on the contract curve but it was obtained by imposing somewhat unrealistic restrictions on the trading possibilities, considering the small number of traders.

### 3. MARKETS WITH MANY TRADERS. THE GAME $(n,n)_0$

Implicitly or explicitly in all models of trading, assumptions have to be made concerning the institutional nature of the market. In the broad sweep of economics, when many individuals are involved, we expect markets to be more or less insensitive to minor institutional differences, but when two individuals engage in face-to-face bargaining we suspect that personality, cultural factors, psychological details, the fine structure of moves and timing, etc., all play a major rôle.

This being the case, it is entirely appropriate to use the "game"  $(1,1)_2$  when we deal with bilateral monopoly. When we discuss markets with many traders on each side, however, we often tend to use something like  $(m, n)_0$  as the model under investigation. In other words, we effectively assume that in markets with many participants on all sides, the individual is constrained to act as a price-taker. There are many good reasons for this, such as the cost of communication, the lack of time to talk to everyone, and other organizational factors that drive toward impersonal mechanisms of trade.

On the other hand, a sociologist or anthropologist might point out that in spite of numerical size and communication problems, patterns can exist and persist in a society that rely on the overt or covert coordination of many individuals. There may be stable configurations

involving the compliance of large groupings or coalitions within the society. Furthermore, both long-run socio-economic considerations and a study of information processes indicate that although many costs and much expenditure of time and energy may be incurred in creating a set of intricate coalitions ab initio, yet given their existence, little effort need be spent in maintaining them. In view of these considerations, it will be worth investigating the characteristic properties of games of the form  $(m, n)_1$  and  $(m, n)_2$ , as well as  $(m, n)_0$ .

For simplicity in the sequel, we shall not only restrict our attention to bilateral markets, but we shall assume  $m = n$ . Traders of the same type are assumed to have identical preferences and initial holdings. Under these assumptions, the discussion of  $(n, n)_0$  becomes especially easy.

Indeed, in going from  $(1, 1)_0$  to  $(n, n)_0$ , the form of the solution remains the same, since the competitive equilibrium point (illustrated for the two-person case as the point  $P$  in Fig. 1) has a direct  $2n$ -dimensional analogue. In fact, the outcome will be the same (under our highly symmetric assumptions) as though  $n$  games of the form  $(1, 1)_0$  were being conducted independently and simultaneously. As  $n$  becomes large the competitive equilibrium point, from the viewpoint of the economist, will be much more "reasonable" as a solution than it was

in the case of bilateral monopoly, for the reasons already given.

#### 4. COOPERATIVE GAMES. THE GAME $(n, n)_2$

##### 4.1. Cores and Solutions

Before we proceed to  $(n, n)_1$  or  $(n, n)_2$ , a brief digression into the subject of cooperative  $n$ -person game theory will be necessary. For ease and clarity the exposition will be mostly in terms of games where the players have transferable, measurable, and comparable utilities. (It is as though the players all attach the same worth to money, and have a constant marginal utility for it.) It must be emphasized that the results we shall give concerning  $(n, n)_2$ , are independent of these utility restrictions.

In order to explain the concepts of cooperative solution and core, we must also define what is meant by the following terms: the characteristic function of a game, an imputation, an effective set of players, and domination of one imputation by another.

The characteristic function specifies the worth that a coalition can achieve if they limit their trades strictly to themselves. Mathematically it is a function  $v(S)$  defined on sets of players  $S$ , with the properties

$$v(\emptyset) = 0,$$

$$v(S \cup T) \geq v(S) + v(T), \text{ whenever } S \cap T = \emptyset.$$

The first condition merely states that the amount achievable by the null set is nothing. The second condition is the fundamental economic property of super-additivity: if two separate groups having commerce only amongst themselves are joined together, the resultant group is at least as effective as were the two independent groups. Beyond these two conditions there is nothing more than can be said a priori about a characteristic function.

If we denote the set of all players in a game by  $N$ , then  $v(N)$  specifies the total amount that the whole group can obtain by cooperation.\* A reasonable form of "cooperative" behavior would be for the players to agree to maximize jointly, and then to decide how the proceeds are to be apportioned, or "imputed." We define an imputation  $\alpha$  to be a division of the proceeds from the jointly optimal play of the game among all the  $n$  players:

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n),$$

where

$$\alpha_i \geq v(i) \text{ and } \sum_{i=1}^n \alpha_i = v(N).$$

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\* Their utilities being transferable, this is properly represented by a single number, which denotes maximum obtainable welfare. If utilities were not transferable,  $v(N)$  would instead have to represent the Pareto-optimal surface, and similarly for smaller coalitions. (See [6], [4].)



The condition  $\alpha_i \geq v(\bar{i})$  embodies the principle that no individual will ever consent to a division that yields him less than he could obtain by acting by himself. It is often convenient to normalize the individual scales so that  $v(\bar{i}) = 0$ .

A set of players is said to be effective for an imputation if by themselves they can obtain at least as much as they are assigned in that imputation. Symbolically,  $S$  is effective for  $\alpha$  if and only if

$$v(S) \geq \sum_{i \in S} \alpha_i .$$

If " $>$ " rather than " $=$ " holds, we shall say that  $S$  is strictly effective.

An imputation  $\alpha$  dominates an imputation  $\beta$  if there exist an effective set  $S$  for  $\alpha$  such that for all members of  $S$ ,  $\alpha_i > \beta_i$ . Following the notation of von Neumann and Morgenstern [9], we write

$$\alpha \succ \beta .$$

In other words, if a set  $S$  of players is in a position to obtain by independent action the amounts that they are offered in the imputation  $\alpha$ , and if, when they compare the amounts offered in  $\alpha$  to the amounts offered in  $\beta$ , all of them prefer the former, then  $\alpha$  dominates  $\beta$ . There is a potential coalition that prefers  $\alpha$  to  $\beta$  and is in a position to do something about it. Note that

S is necessarily strictly effective for  $\beta$ , the dominated imputation.

Finally, we may define two "solution" concepts. The core of an n-person game is the set of undominated imputations, if any. A von Neumann-Morgenstern solution, on the other hand, consists of a set of imputations which do not dominate each other, but which collectively dominate all alternative imputations. There is at most one core, but there may be many solutions. All solutions contain the core, if it exists.

#### 4.2. Some Examples

A series of simple, three-person games will illustrate these concepts. Consider first the game in which any player acting by himself obtains nothing, but any pair of players acting in concert can demand three units to share between them, while all three players in coalition are also awarded three. The characteristic function of this game is

$$v(\emptyset) = 0,$$

$$v(\overline{1}) = v(\overline{2}) = v(\overline{3}) = 0,$$

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 3,$$

$$v(\overline{123}) = 3,$$

where  $\overline{12}$  means "the set consisting of players 1 and 2," etc.

We may represent the imputations in this game by triangular coordinates, as shown in Fig. 2. The vertices  $P_1, P_2, P_3$  represent the imputations  $(3, 0, 0), (0, 3, 0)$ , and  $(0, 0, 3)$ , respectively. The point  $\omega = (1, 1, 1)$  is the center of the triangle. Consider the two imputations  $\alpha = (1.9, 0, 1.1)$  and  $\beta = (0, 1.5, 1.5)$ . The set  $\overline{23}$  is effective for  $\beta$ , and furthermore both 2 and 3 are better off in  $\beta$  than in  $\alpha$ . Hence  $\beta \succsim \alpha$ .

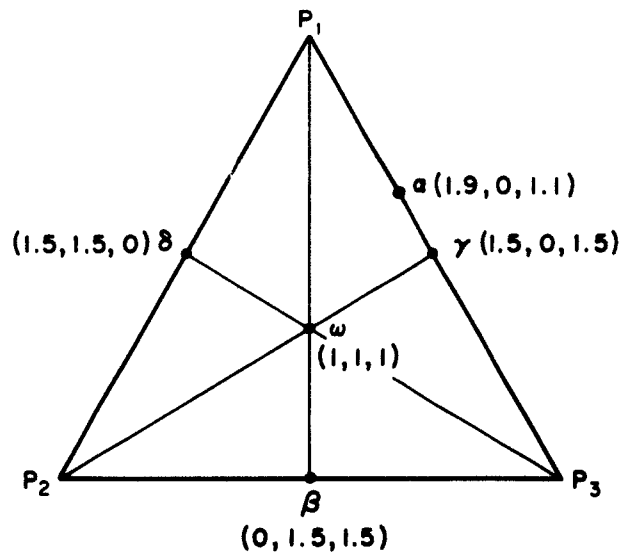


Fig. 2

The trio of imputations  $\beta$ ,  $\gamma$ , and  $\delta$  forms a solution set to this particular game. Any other imputation gives two of the players less than 1.5 apiece, and thus is dominated by one of these three imputations, but the three do not dominate each other. (There are other solution sets, which we need not discuss.) This game has no core, since the imputations  $\beta$ ,  $\gamma$ , and  $\delta$ , dominating all the rest, are themselves dominated by others. For example, the imputation  $\alpha$ , which was dominated by  $\beta$ , in turn dominates  $\delta$  via the effective set  $\overline{13}$ . Note that domination is not a transitive relation:  $\beta \succ \alpha$  and  $\alpha \succ \delta$  do not entail  $\beta \succ \delta$ .

We now consider three closely related games, differing from the previous one only in what the two-person coalitions obtain. In the first variant we have

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 0.*$$

In this case, all imputations are in the core. The only

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\* This all-or-nothing type of characteristic function, like the previous one, is associated more with political than economic processes [10]. The previous game was a majority-take-all situation; the present one is a veto situation, since if one member wishes to be the "dog in the manger," he can prevent the others from obtaining any payoff. In economics such extremes—called simple games—are not typical. We shall presently consider variants in which the two-person coalitions obtain intermediate amounts, reflecting the more usual situation in which any new adherent to a coalition means added possibilities for profit.

set of players that is effective, for most imputations, is the three-person set; however, this is useless for domination, since on examining the distribution of welfare from the viewpoint of all three players we see that if one player prefers one of two imputations, then at least one of the other players will prefer the other, the sum of the allotments being constant. In fact, it suffices to point out that no set of players is strictly effective for any imputation—hence there is no domination. The core is therefore as large as possible, and is also the unique von Neumann-Morgenstern solution.

In our third example we assume

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 2.$$

As shown in Fig. 3, the lines which describe the amount obtainable by each coalition of two players intersect in a single point, the imputation  $w$  with coordinates  $(1, 1, 1)$ . This is the only undominated imputation of the game, and thus constitutes a single-point core. Since  $w$  fails to dominate the three small triangles adjoining it in the diagram, however, it is not a von Neumann-Morgenstern solution by itself. To get a solution we must add some more or less arbitrary curves, as shown, traversing the three triangular regions (see [9], pp. 550-554).

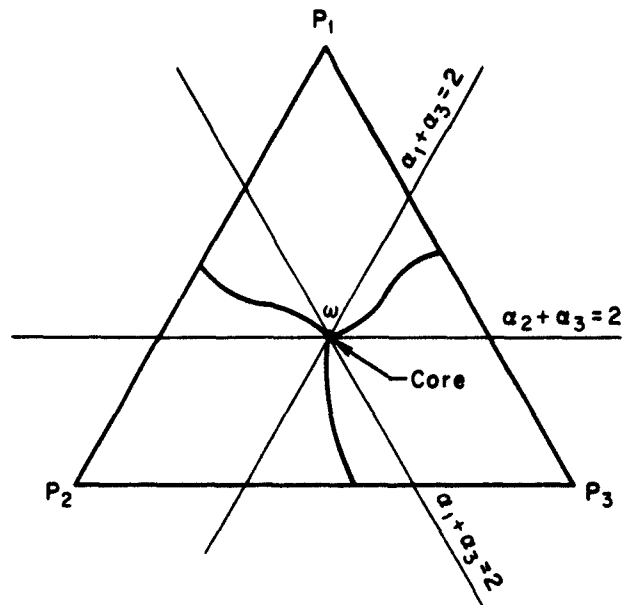


Fig. 3

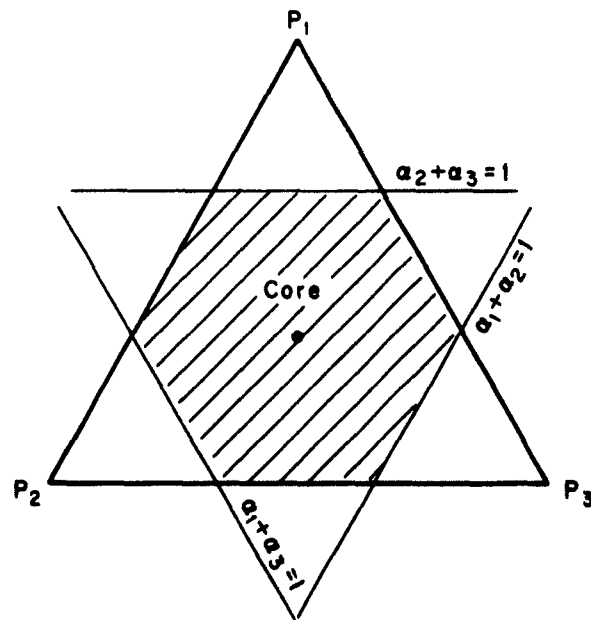


Fig. 4

In the final variant, we assume that the two-person coalitions are only half as profitable as in the preceding example. That is, we have

$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 1.$$

The lines indicating the ranges of effectiveness of these coalitions are spread apart, as shown in Fig. 4, revealing a large, hexagonal core. All imputations in that area are undominated. As in the second example, this core is the unique solution.

A superficial examination of these four examples suggests a relationship between the size of the core and the "fatness" of the coalitions in a game, i.e., how much they can promise their members per capita as compared to the per-capita amount available in the whole game. In all four instances, the latter amount was  $v(\overline{123})/3 = 1$ . Denote  $v(\overline{ij})/2$  by  $f_2$ . In the first game,  $f_2$  was 1.5, which is greater than 1, and there was no core. In the third game,  $f_2$  was exactly 1, and the core was a single point. In the fourth game,  $f_2$  was  $1/2$ , and there was a large core, while in the second game  $f_2$  was 0, and every imputation was in the core.

Of course, in a less symmetric situation, this principle would not reveal itself in such a clean-cut manner. However, a general rule of thumb seems to persist: the more power there is in the hands of the middle-sized

groups, the more narrowly circumscribed is the range of outcomes of the cooperative game. This rule, vague as it is, applies to solutions as well as cores.

When we do not permit the transfer of utility, we can no longer talk about the amount attainable by a group of players as a single number; nevertheless cores [6] and solutions [11] can still be defined, and the idea of the per-capita gain in one coalition being larger than in another can still be utilized in a vectorial sense.

#### 4.3. The Edgeworth Market Game, $(n, n)_2$

Let us return to the Edgeworth bilateral-monopoly game  $(1, 1)_2$ . Even without the simplifying assumption of transferability of utility, it is not hard to see that the Edgeworth contract curve is the "cooperative solution," in the spirit of the von Neumann-Morgenstern definition but with weaker utility assumptions [11]. It is also the core. The reason is simply that with just two players, there are no coalitions of intermediate size, between the individual and the whole group, and hence no domination occurs between outcomes that are both Pareto optimal and individually rational.

It can be shown that the whole contract curve (considered in the higher dimensions) will remain as a cooperative solution to the game  $(n, n)_2$  for any number of traders. However, it will no longer be the core if



there are more than two players, and there will be many other von Neumann-Morgenstern-type solutions, which may in general be quite difficult to compute (see [4]).

It has been shown that as the number of players increases, the core shrinks down upon the competitive equilibrium point (or set of points) [3, 5, 12]. Hence, if we regard the core as our concept of cooperative solution, the game  $(1, 1)_2$  has the contract curve as its solution, while the game  $(n, n)_2$ , as  $n$  grows large, has an increasingly determinate solution consisting of a small neighborhood of the competitive equilibrium.

Figure 5 illustrates the shrinking of the core as the number of players is increased. It is similar to Fig. 1 of Edgeworth ([1], pp. 20-25). The line  $RP$  is the competitive price ray on the exchange diagram for  $2n$  traders, consisting of  $n$  of each type. The arc  $CC'$  is the range of the two-person contract curve, and  $I$  is a typical point on that curve. The shaded area between  $I$  and  $I_1$  indicates a domain that is preferred to  $I$  by a player of the first type; similarly the shaded area between  $I$  and  $I_2$  is preferred to  $I$  by a player of the second type. These areas are bounded by the two indifference curves which are tangent at  $I$ ; each area includes a portion of the price ray  $RQ$  only if  $I$  is not the competitive allocation  $P$ . We shall describe

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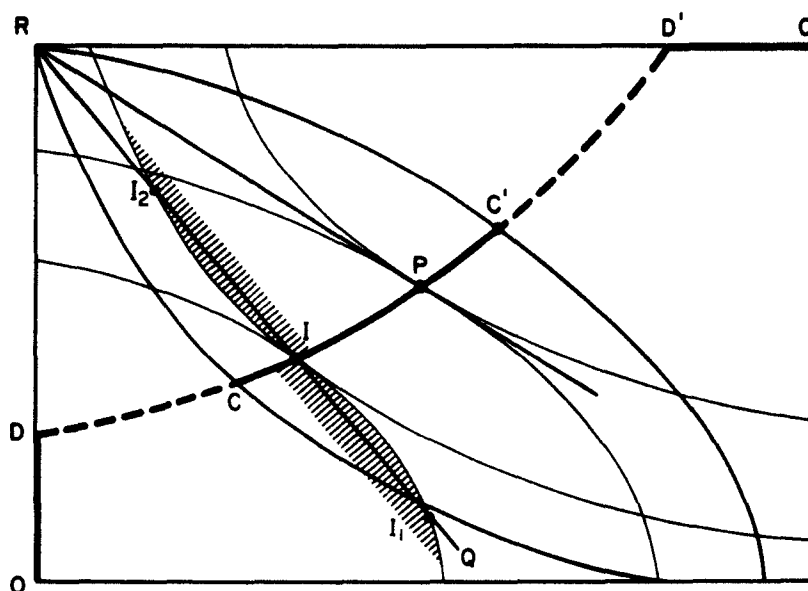


Fig. 5

the conditions under which  $I$  (or more precisely, the imputation associated with  $I$ ) is dominated by the outcome of some trade among a subset of the players.

When only two traders enter into contract, exchanging  $x$  units of the first good for  $y$  of the second, the final distribution of assets between them is given by  $(a-x, y)$  and  $(x, b-y)$ . This can be represented by a single point on the Edgeworth diagram. Suppose that a coalition of  $k$  traders of the first type and  $m$  traders of the second type forms, and suppose that traders of the same type decide to divide their gains equally. Together the

coalition controls  $ka$  units of the first commodity and  $mb$  of the second. After trade, a player of the first type will have  $(a - \frac{x}{k}, \frac{y}{k})$ , and a player of the second type will have  $(\frac{x}{m}, b - \frac{y}{m})$ , for some  $x, y$ . This outcome can be represented on the Edgeworth diagram by two points, corresponding to the two types of players, which fall on the line  $RQ$ , one on either side of the contract curve. If  $k = m$ , the two points coincide at  $I$ . If the ratio of  $k$  to  $m$  is only slightly different from unity, it will be possible for the point corresponding to each type of player to lie in the shaded region in which that type of player gains, in comparison to the distribution offered at  $I$ . The imputation corresponding to  $I$  will therefore be dominated. As  $n$  grows larger, the available ratios  $k/m$  become denser around 1, and the portion of the contract curve that escapes domination shrinks down to an arbitrarily small neighborhood of the competitive equilibrium point. (If the latter is not unique, the convergence will be to the set of competitive equilibria.)

It is remarkable that such widely different sets of modeling assumptions as were used in  $(n, n)_0$  and  $(n, n)_2$  should lead, in the limit, to the same solution. In the one case, the participants operate on a minimum of information, they have essentially no strategic initiative, and they are prevented from cooperating or even interacting with

their fellows. In the other case, information is freely available, exchanges of goods may be made in all possible ways, without regard for prices, and collusion not only is permitted, but is in fact essential to the maintenance of stability.

Viewed as a limit of cores, the competitive equilibrium for large  $n$  is seen to be "sociologically neutral." No coalition is effective against it. It is not a von Neumann-Morgenstern-type solution by itself, even in the limit, since it does not dominate the other points on the contract curve; nevertheless, every such solution will include the competitive imputation. Thus, without any dynamic assumptions regarding prices or other mechanisms of the market place, and without special hypotheses concerning costs of communication, information, and so forth, the competitive equilibrium plays an important and distinctive rôle purely on the grounds of social stability.

## 5. NONCOOPERATIVE GAMES. THE COURNOT OLIGOPOLY GAME

### 5.1. Noncooperative Equilibrium Points

An approach highly different from that of cooperative game theory, and connected less directly to welfare considerations than to problems of oligopoly and control of industry, is the idea of the noncooperative solution to a game. The spirit of much of the classical discussion of oligopoly behavior and equilibrium in monopolistic

competition has been to regard the firms as powerful and their customers as weak. This conforms to our commonsense notions of the oligopolistic market place. The automobile or tobacco companies, for example, have some form of control over output, prices, brands, and so forth, but their customers are, for the most part, price-takers.

Both the classical and many of the more recent works in oligopoly theory have made use of "open" models, in the sense that they investigate the behavior of a group of firms or an industry, taking the behavior of customers and suppliers as given. Underlying the writings of Cournot [13], Bertrand [14], Edgeworth [15], Hotelling [16], Stackelberg [17], Chamberlin [18], and others, is the concept of the attainment of a noncooperative equilibrium by the firms in competition. On the mathematical side, Nash [19] was first to develop the basic idea of noncooperative solution in the abstract framework of the theory of games. The crucial concept in the noncooperative theory is that of "equilibrium point," which may be defined as follows:

Consider a game with  $n$  players. Let player  $i$  have a class of possible "strategies"  $S_i$ , and let  $s_i$  denote a particular strategy belonging to the class. The payoff to the  $i$ -th player is denoted by  $P_i(s_1, s_2, s_3, \dots, s_n)$ , a function of the strategies of all the players. A strategy vector  $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \dots, \bar{s}_n)$  is said to constitute an equilibrium point if, for all  $i$ , the function

$$P_1(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_{i-1}, s_i, \bar{s}_{i+1}, \dots, \bar{s}_n)$$

is maximized by setting  $s_i = \bar{s}_i$ . In other words, a set of strategies, one for each player, forms an equilibrium point if each player, knowing the strategies of all others, will not be motivated to change. If the equilibrium point is unique, as it proves to be in many of the classical economic models, it is termed the "noncooperative solution" to the game.

## 5.2. The Cournot Model

An early economic example of noncooperative equilibrium was presented by Cournot [13]. Suppose that two firms are constrained to name amounts  $q_1, q_2$  offered for sale. These quantities are their strategic variables. Furthermore suppose that there is a market mechanism of some sort which selects a price that exactly clears the market, say  $p = D(q_1 + q_2)$ . Let the cost functions of the two firms be  $C_1(q_1)$  and  $C_2(q_2)$ , respectively. The payoffs are then

$$P_1 = q_1 D(q_1 + q_2) - C_1(q_1),$$

$$P_2 = q_2 D(q_1 + q_2) - C_2(q_2),$$

respectively. An equilibrium point will be a pair of strategies  $(\bar{q}_1, \bar{q}_2)$  such that

$P_1(q_1, \bar{q}_2)$  is maximized at  $q_1 = \bar{q}_1$  ,

$P_2(\bar{q}_1, q_2)$  is maximized at  $q_2 = \bar{q}_2$  .

This is the "Cournot" solution [20].

In the game described above, the welfare of the customers was included only implicitly, through the workings of the demand relation; it is thus an open model. Considering only the welfare of the two duopolists, it is easy to see that the noncooperative equilibrium is not Pareto optimal in the open model. They could both improve matters by joint action, restricting the amounts offered to be sold.

It is nevertheless conceivable that the noncooperative equilibrium might be Pareto optimal in the closed model, in which the welfares of the customers are also taken into account; however, even this is not the case in general. We already have a simple example of this in the game  $(1, 1)_1$ , which can be viewed as a monopolist dealing with a compliant, price-taking customer. In this case, quantity naming is usually equivalent to price naming. Referring to Fig. 6, we see that if the monopolist (origin at  $O'$ ) offers the amount  $e$  to be sold, for example, the price mechanism will determine the price ray  $RQ$  that just clears the market when the other player optimizes, selecting the point  $M$ , where the ray intersects his response curve  $ST$ . Under suitable assumptions, this



derived price ray will be unique, for any value of  $e$ , giving the monopolist effective control over prices.

Returning to our point about optimality, we note that the best outcome from the viewpoint of the monopolist is the point  $M_1$ , which is clearly not Pareto optimal in the closed model since it lies off the contract curve. In this instance of monopolistic exploitation, all could have their welfare improved by negotiation.

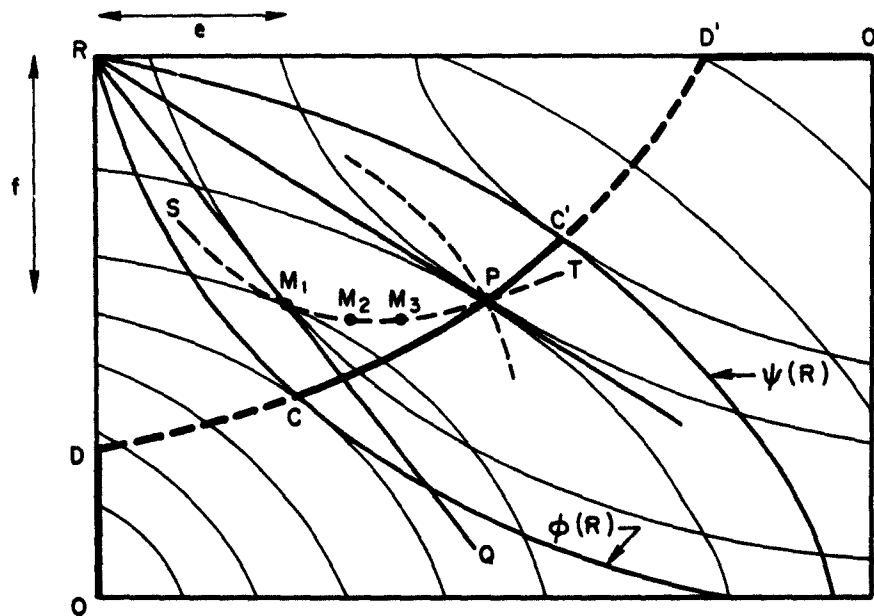


Fig. 6

### 5.3. The Game $(n, n)_1$

In order to define the game  $(n, n)_1$  we must specify the market mechanism more fully, making explicit the scope of strategic choice. There are many ways in which this can be done. We might assume that the firms playing the rôle of oligopolists are in a position to name price. This is the manner in which Bertrand and Edgeworth handled oligopoly. Though complicated, it is possible to specify a mechanism whereby each oligopolist names both a price and a limit to the amount he is willing to trade. In order to complete the specification, the nature of the market-clearing process, in the presence of possibly different prices, would also have to be made explicit.

An easier model to define, involving fewer arbitrary institutional assumptions, is the analogue to Cournot's approach, in which the oligopolists are free to specify the amounts to be offered for trade, but not the prices. A mechanism then sums together all the offers and calculates the prices that will exactly clear the market. The traders of the second type are constrained to be price-takers, as before. We shall adopt this quantity-naming model as the most natural generalization of the game  $(1, 1)_1$ , in which, as already noted, price naming and quantity naming are essentially equivalent.

Let us assume that the "response curve" (ST in Fig. 6) of the second player in the two-person game  $(1, 1)_1$  can be expressed as a single-valued function  $y = r(x)$ , where  $(x, y)$  is that player's final holding, after trade. In addition, we shall assume that  $r(x)$  has a bounded derivative. (The purpose of these assumptions is to exclude certain difficulties related to the possibility of different prices yielding the same demand.) Then, in the game  $(n, n)_1$ , the price that clears the market is given by

$$p = \frac{b - r(\bar{q})}{\bar{q}},$$

where  $\bar{q} = \sum_i q_i / n$  is the average quantity offered of the first good. The final holdings will be

$$(a - q_i, pq_i), \quad i = 1, 2, \dots, n,$$

for the players of the first type, and

$$(\bar{q}, b - p\bar{q})$$

for each player of the second type. The noncooperative equilibrium is found by requiring the function

$V(a - q_i, pq_i)$  to be a maximum with respect to  $q_i$ , for each  $i$ .

This maximum will generally not be found at the point  $M_1$  in Fig. 6, where  $r(x)$  is tangent to the

family  $\gamma$ . The reason is that when a player varies his offer  $q_i$ , the average offer  $\bar{q}$ , which is what determines the price, varies by only  $1/n$  as much. In fact, the oligopolists will tend to offer larger quantities than they would if they were in monopolistic collusion, and the noncooperative equilibrium outcome  $M_n$  will be displaced along the response curve in the direction of the competitive equilibrium outcome  $P$ . (See Fig. 6.)

To see why  $M_n$  converges to  $P$  in the limit, we merely set  $\partial \Psi(a - q_i, pq_i) / \partial q_i = 0$ , and obtain

$$p = \frac{\gamma x}{\gamma y} - q_i \frac{\partial p}{\partial q_i} = \frac{\gamma x}{\gamma y} + \frac{q_i}{n\bar{q}} (r' + p),$$

a relation which must hold at  $M_n$ . But  $r'$  is bounded by assumption, as is  $q_i$ , and it can be shown that  $\bar{q}$  does not go to zero; therefore

$$p \rightarrow \frac{\gamma x}{\gamma y} \text{ as } n \rightarrow \infty.$$

This implies that  $M_n$  lies on the response curve of the players of the first type as well as of those of the second type. Hence  $M_n = P$ .\*

The intuitive idea behind this argument is that when  $n$  becomes large, the effect of one individual on the price structure becomes negligible; i.e.,  $\partial p / \partial q_i \rightarrow 0$ .

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\*More precisely, since uniqueness is not assured, any limit point of any sequence  $\{M_n\}$  of noncooperative equilibria is a competitive allocation. Our assumptions regarding  $r(x)$  are essential to this result.

## 6. CONCLUSION

We began with three different models of bilateral monopoly, based on the same data. Three different solution concepts were advanced, each appropriate to one of the models, and three qualitatively different outcomes were predicted. As the number of traders on both sides of the market was increased, the three solutions merged into one as regards predicted outcome, but their rationales remained quite distinct. A relationship was thereby demonstrated between (1) administered price stability (the competitive equilibrium), (2) noncollusive oligopolistic exploitation (the equilibrium point of a noncooperative game), and (3) unrestricted bargaining between coalitions (the core of a cooperative game).

Each model embodied radically different assumptions concerning the strategies and information available to the participants. A scrutiny of the differences would reveal many places where sociological and institutional assumptions might be slipped in, perhaps inadvertently, in the construction of models of markets.

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